arXiv:1101.3479v1 [math.CV] 18 Jan 2011

ON THE PACKING DIMENSION OF THE JULIA SET AND THE ESCAPING SET OF AN ENTIRE FUNCTION

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Abstract. Let f be a transcendental entire function. We give conditions which imply that the Julia set and the escaping set of f have packing dimension 2. For example, this holds if there exists a positive constant c less than 1 such that the minimum modulus $L(r,f)$ and the maximum modulus $M(r,f)$ satisfy $\log L(r,f) \leq c \log M(r,f)$ for large r. The conditions are also satisfied if $\log M(2r,f) \geq d \log M(r,f)$ for some constant d greater than 1 and all large r.

1. Introduction and results

The Fatou set $F(f)$ of an entire function f is defined as the set of all $z \in \mathbb{C}$ where the iterates f^n of f form a normal family. The Julia set is the complement of $F(f)$ and denoted by $J(f)$. The escaping set $I(f)$ is the set of all $z \in \mathbb{C}$ for which $f^n(z) \to \infty$ as $n \to \infty$. We note that $J(f) = \partial I(f)$ by a result of Eremenko [14]. For an introduction to the iteration theory of transcendental entire functions we refer to [5].

Considerable attention has been paid to the dimensions of Julia sets of entire functions; see [36] for a survey, as well as [3, 4, 8, 9, 10, 27, 28, 34] for some recent results not covered there. Many results in this area are concerned with the Eremenko-Lyubich class B consisting of all transcendental entire functions for which the set of critical and finite asymptotic values is bounded. By a result of Eremenko and Lyubich [15, Theorem 1] we have $I(f) \subset J(f)$ for $f \in B$. For a function in the Eremenko-Lyubich class, a lower bound for the dimension of the Julia set can thus be obtained from such a bound for the escaping set. This played a key role already in McMullen's seminal paper [24], and it has been used in many subsequent papers.

We denote the Hausdorff dimension, packing dimension and upper box dimension of a subset A of the complex plane $\mathbb{C}$ by $\dim_{\mathrm{H}} A$, $\dim_{\mathrm{P}} A$ and $\overline{\dim}_{\mathrm{B}} A$, respectively, noting that the upper box dimension is defined only for bounded sets A. We refer to the book by Falconer [16] for the definitions and a thorough treatment of these concepts. Here we only note that we always have [16, p. 48]

$$\dim_{\mathrm{H}} A \leq \dim_{\mathrm{P}} A \leq \overline{\dim}_{\mathrm{B}} A.$$

1991 *Mathematics Subject Classification.* Primary 37F10; Secondary 30D05, 30D15, 37F35.

Supported by a Chinese Academy of Sciences Visiting Professorship for Senior International Scientists, Grant No. 2010 TIJ10. Also supported by the Deutsche Forschungsgemeinschaft, Grant Be 1508/7-1, the EU Research Training Network CODY and the ESF Networking Programme HCAA.

The exceptional set $E(f)$ of a transcendental entire function f consists of all points in $\mathbb{C}$ with finite backward orbit. It is an immediate consequence of Picard's theorem that $E(f)$ contains at most one point. The following result is one part of a theorem of Rippon and Stallard [30, Theorem 1.2].

Theorem A. *Let f be a transcendental entire function, A a backward invariant subset of $J(f)$ and U a bounded open subset of $\mathbb{C}$ whose closure does not intersect $E(f)$. Then $\overline{\dim}_{\mathrm{B}}(U \cap A) = \dim_{\mathrm{P}} A$. In particular, $\overline{\dim}_{\mathrm{B}}(U \cap J(f)) = \dim_{\mathrm{P}} J(f)$.*

For functions in the Eremenko-Lyubich class they obtained the following result [30, Theorem 1.1].

Theorem B. *Let $f \in B$. Then $\dim_{\mathrm{P}} J(f) = 2$.*

It follows from Theorem A that Theorem B is equivalent to the result that $\overline{\dim}_{\mathrm{B}}(U \cap J(f)) = 2$ for some bounded open set U satisfying $\overline{U} \cap E(f) = \emptyset$. In order to show this, Rippon and Stallard actually proved that $\overline{\dim}_{\mathrm{B}}(U \cap I(f)) = 2$ for such a set U and then used the result of Eremenko and Lyubich quoted above.

The main tool used by Eremenko and Lyubich to prove this result is a logarithmic change of variable. This method shows in particular that if $f \in B$, then f is bounded on a curve tending to ∞; see [15, p. 993]. The $\cos \pi\rho$-theorem (see [19, Chapter 5, Theorem 3.4] or [21, Section 6.2]) now implies that

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{\sqrt{r}} > 0, \tag{1.1}$$

where

$$M(r,f) := \max_{|z|=r} |f(z)|$$

is the maximum modulus. (The observation that (1.1) holds for functions in B seems to have been made first in [7, Proof of Corollary 2] and [23, p. 1788]; see also [30, Lemma 3.5].)

It follows from a result of Baker [2, Corollary to Theorem 3.1] that all components of $F(f)$ are simply connected if f is bounded on a curve tending to ∞. In particular, if $f \in B$, then $F(f)$ has no multiply connected components [15, Proposition 3].

In view of these results the following theorem can be considered as a generalization of Theorem B.

Theorem 1.1. *Let f be a transcendental entire function satisfying*

$$\liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log\log r} = \infty. \tag{1.2}$$

If $F(f)$ has no multiply connected component, then

$$\dim_{\mathrm{P}}(I(f) \cap J(f)) = 2. \tag{1.3}$$

Since multiply connected components of $F(f)$ are contained in $I(f)$, we see that $I(f)$ has interior points if $F(f)$ has such a component. We conclude that $\dim_P I(f) = 2$ for all entire functions satisfying (1.2).

The following result is an immediate consequence of Theorem 1.1 and the results stated before it.

Corollary 1.1. *Let f be a transcendental entire function which is bounded on a curve tending to ∞. Then* $\dim_{\mathrm{P}}(I(f) \cap J(f)) = 2$.

More generally, we have the following result involving the minimum modulus

$$L(r, f) := \min_{|z|=r} |f(z)|.$$

Corollary 1.2. *Let f be a transcendental entire function and suppose that*

$$\limsup_{r\to\infty} \frac{\log L(r, f)}{\log M(r, f)} < 1. \tag{1.4}$$

Then $\dim_{\mathrm{P}}(I(f) \cap J(f)) = 2$.

To deduce Corollary 1.2 from Theorem 1.1 we note that the $\cos \pi\rho$-theorem yields that

$$\liminf_{r\to\infty} \frac{\log \log M(r, f)}{\log r} > 0 \tag{1.5}$$

under the hypothesis (1.4). Clearly (1.2) follows from (1.5).

Zheng [37, Corollary 1] proved, as a corollary to the main result of his paper, that if $F(f)$ has a multiply connected component, then

$$\limsup_{r\to\infty} \frac{\log L(r, f)}{\log M(r, f)} > 0. \tag{1.6}$$

A slight extension of his argument shows that his main result actually yields that

$$\limsup_{r\to\infty} \frac{\log L(r, f)}{\log M(r, f)} = 1 \tag{1.7}$$

if $F(f)$ has a multiply connected component; see Proposition 4.1. Thus the hypotheses of Theorem 1.1 are satisfied if (1.4) holds. Actually the condition (1.4) can be further relaxed; cf. Remark 4.2.

Zheng [37, Corollary 5] also showed that the Fatou set of a transcendental entire function f has no multiply connected component if

$$\log M(2r, f) \geq d \log M(r, f) \tag{1.8}$$

for some $d > 1$ and all large r. It is easy to see that (1.8) implies (1.5) and hence (1.2). Thus we obtain the following corollary to Theorem 1.1.

Corollary 1.3. *Let f be a transcendental entire function satisfying* (1.8). *Then* $\dim_{\mathrm{P}}(I(f) \cap J(f)) = 2$.

We mention that in Theorem B and in Theorem 1.1, as well as in the corollaries to Theorem 1.1, the packing dimension cannot be replaced by the Hausdorff dimension. In fact, it is shown in [28, Corollary 1.4] that there exists a function $f \in B$ for which $\dim_{\mathrm{H}} I(f) = 1$ and in [35] that for every $\varepsilon > 0$ there exists a function $f \in B$ such that $\dim_{\mathrm{H}} J(f) < 1 + \varepsilon$, and the functions considered in [28, 35] satisfy (1.8) as well. On the other hand, for every transcendental entire function f the Hausdorff dimension of $I(f) \cap J(f)$ is at least 1, since this set contains continua; see [29, Theorem 5] and [33, Theorem 1.3].

Concerning the proof of Theorem 1.1, we note that in view of Theorem A the conclusion of Theorem 1.1 is equivalent to the statement that

$$\overline{\dim}_{\mathrm{B}}(U \cap I(f) \cap J(f)) = 2 \tag{1.9}$$

for some bounded open set U satisfying $\overline{U} \cap E(f) = \emptyset$, which in turn is equivalent to (1.9) holding for all bounded open sets U. We shall show in our proof that (1.9) holds for some bounded open set U whose closure does not intersect $E(f)$.

The main tools used in the proof are certain estimates of the logarithmic derivative and a version of the Ahlfors islands theorem. A similar technique was used in [8] where it was shown that under a suitable regularity condition on the growth of f we even have $\dim_{\mathrm{H}}(I(f) \cap J(f)) = 2$.

2. Preliminary lemmas

We use the standard terminology of Nevanlinna theory and, in particular, denote by $T(r, f)$ the Nevanlinna characteristic of a meromorphic function f; see [19, 20]. First we note that using the well-known inequality

$$T(r, f) \leq \log^+ M(r, f) \leq \frac{R + r}{R - r} T(R, f) \quad \text{for } 0 < r < R \tag{2.1}$$

we easily see that the growth condition (1.2) is equivalent to

$$\liminf_{r \to \infty} \frac{\log T(r, f)}{\log \log r} = \infty. \tag{2.2}$$

We shall need a number of lemmas and begin with the following estimate of the logarithmic derivative [19, p. 88].

Lemma 2.1. *Let f be an entire function satisfying $f(0) = 1$. Then*

$$\left| \frac{f'(z)}{f(z)} \right| \leq \frac{4s}{(s - |z|)^2} T(s, f) + \sum_{|z_j| \leq s} \frac{2}{|z - z_j|},$$

for $s > |z|$, where (z_j) is the sequence of zeros of f.

In order to estimate the sum on the right hand side we shall use the following result due to Fuchs and Macintyre [18]. Here and in the following we denote by $D(a, r)$ the open disk and by $\overline{D}(a, r)$ the closed disk of radius r around a point a.

Lemma 2.2. *Let $z_1, z_2, \dots, z_n \in \mathbb{C}$ and let $H > 0$.*

(i) *There exist $l \in \{1, 2, \dots, n\}$, $u_1, u_2, \dots, u_l \in \mathbb{C}$ and $s_1, s_2, \dots, s_l > 0$ satisfying*

$$\sum_{k=1}^{l} s_k^2 \leq 4H^2$$

such that

$$\sum_{k=1}^{m} \frac{1}{|z - z_k|} \leq \frac{2n}{H} \quad \text{for } z \notin \bigcup_{k=1}^{l} D(u_k, s_k).$$

(ii) *There exist* $m \in \{1, 2, \ldots, n\}$, $v_1, v_2, \ldots, v_m \in \mathbb{C}$ *and* $t_1, t_2, \ldots, t_m > 0$ *satisfying*

$$\sum_{k=1}^{m} t_k \le 2H$$

such that

$$\sum_{k=1}^{n} \frac{1}{|z - z_k|} \le \frac{n(1 + \log n)}{H} \quad \text{for } z \notin \bigcup_{k=1}^{m} D(v_k, t_k).$$

Actually Fuchs and Macintyre write An/H instead of $2n/H$ in part (i), with a constant A, but their argument shows that one can take $A = 2$. We note that the term $\log n$ in (ii) cannot be omitted; cf. [1].

We will also require the following version of the Borel-Nevanlinna growth lemma; see [19, Chapter 3, Theorem 1.2] or [12, Section 3.3]. Here a measurable subset E of $(0, \infty)$ is said to be of finite logarithmic measure if

$$\int_E \frac{dt}{t} < \infty.$$

Lemma 2.3. *Let* $F : [r_0, \infty) \to [t_0, \infty)$ *and* $\varphi : [t_0, \infty) \to (0, \infty)$ *be non-decreasing functions, with* $r_0, t_0 > 0$. *Suppose that*

$$\int_{t_0}^{\infty} \frac{dt}{\varphi(t)} < \infty.$$

Then there exists a set $E \subset [r_0, \infty)$ *of finite logarithmic measure such that*

$$F\left(r\left(1 + \frac{1}{\varphi(F(r))}\right)\right) \le F(r) + 1 \quad \text{for } r \notin E.$$

We shall also need a result from the Ahlfors theory of covering surfaces; cf. [20, Theorem 6.2] or [6]. Here $\mathbb{D} := D(0, 1)$ is the unit disk.

Lemma 2.4. *Let* D_1, D_2, D_3 *be Jordan domains with pairwise disjoint closures. Then there exists* $\mu > 0$ *such that if* $h : \mathbb{D} \to \mathbb{C}$ *is a holomorphic function satisfying*

$$\frac{|h'(0)|}{1 + |h(0)|^2} \ge \mu,$$

then $\mathbb{D}$ *has a subdomain which is mapped bijectively onto one of the domains* D_ν *by* h.

Finally, we shall repeatedly use the following result known as the Koebe distortion theorem and the Koebe one quarter theorem.

Lemma 2.5. *Let* $g : D(a, r) \to \mathbb{C}$ *be univalent,* $0 < \lambda < 1$ *and* $z \in \overline{D}(a, \lambda r)$. *Then*

$$\frac{1}{(1+\lambda)^2}|g'(a)| \le \frac{|g(z) - g(a)|}{|z - a|} \le \frac{1}{(1-\lambda)^2}|g'(a)|$$

and

$$\frac{1 - \lambda}{(1+\lambda)^3}|g'(a)| \le |g'(z)| \le \frac{1 + \lambda}{(1-\lambda)^3}|g'(a)|.$$

Moreover,

$$g(D(a,r)) \supset D\left(g(a), \frac{1}{4}|g'(a)|r\right).$$

Usually Koebe's theorems are stated only for the special case that $a = 0$, $r = 1$, $g(0) = 0$ and $g'(0) = 1$, but the above result easily follows from this special case.

3. Proof of Theorem 1.1

Without loss of generality we may assume that $f(0) = 1$. We denote by $n(r, 0)$ the number of zeros of f in the closed disk of radius r around 0 and put

$$N(r,0) := \int_0^r \frac{n(t,0)}{t} dt.$$

By Nevanlinna's first fundamental theorem we have

$$T(s,f) \geq N(s,0) \geq \int_r^s \frac{n(t,0)}{t} dt \geq n(r,0) \int_r^s \frac{dt}{t} = n(r,0) \log\left(\frac{s}{r}\right) \tag{3.1}$$

for $s > r > 0$. Applying Lemma 2.3 with $\varphi(x) = x^2/6$ and $F(r) = \log T(r,f)$ we obtain a set E of finite logarithmic measure such that

$$T\left(r\left(1 + \frac{6}{[\log T(r,f)]^2}\right), f\right) \leq eT(r,f) \quad \text{for } r \notin E. \tag{3.2}$$

For $m \in \mathbb{N}$ we shall use the abbreviation

$$R_m(r) := r\left(1 + \frac{m}{[\log T(r,f)]^2}\right)$$

so that (3.2) takes the form

$$T\left(R_6(r), f\right) \leq eT(r,f) \quad \text{for } r \notin E. \tag{3.3}$$

Using (2.1) we find that if $r \notin E$ is sufficiently large, then

$$\log M(r,f) \leq \frac{R_6(r) + r}{R_6(r) - r} T(R_6(r), f) \leq T(r,f)[\log T(r,f)]^2. \tag{3.4}$$

For measurable $X \subset \mathbb{R}$ and $Y \subset \mathbb{C}$ we denote by $\operatorname{length} X$ the 1-dimensional Lebesgue measure of X and by $\operatorname{area} Y$ the 2-dimensional Lebesgue measure of Y.

Lemma 3.1. *For sufficiently large $r \notin E$ there exists a closed subset F_r of $[R_1(r), R_3(r)]$ with*

$$\operatorname{length} F_r \geq \frac{r}{[\log T(r,f)]^2} \tag{3.5}$$

such that

$$\left|\frac{f'(z)}{f(z)}\right| \leq \frac{T(r,f)[\log T(r,f)]^7}{r} \quad \text{for } |z| \in F_r. \tag{3.6}$$

Moreover, for $\eta > 0$ there exist an integer $l \in \{1, 2, \ldots, n\left(R_5(r), 0\right)\}$, points $u_1, u_2, \ldots, u_l \in \mathbb{C}$ and $s_1, s_2, \ldots, s_l > 0$ satisfying

$$\sum_{k=1}^{l} s_k^2 \leq \frac{r^2}{T(r,f)^\eta} \tag{3.7}$$

such that

$$\left|\frac{f'(z)}{f(z)}\right| \leq \frac{T(r,f)^{1+\eta}}{r} \quad \text{if } r \leq |z| \leq R_4(r) \text{ and } z \notin \bigcup_{k=1}^{l} D(u_k, s_k). \tag{3.8}$$

Proof. It follows from Lemma 2.1 that if $z_1, z_2, \dots$ are the zeros of f, then

$$\left|\frac{f'(z)}{f(z)}\right| \leq \frac{4R_5(r)}{(R_5(r)-|z|)^2}T(R_5(r),f) + \sum_{|z_j|\leq R_5(r)} \frac{2}{|z-z_j|} \tag{3.9}$$

for $|z| < R_5(r)$. Using (3.3) and noting that $R_5(r) \leq 2r$ for large r we see that if $|z| \leq R_4(r)$, then

$$\frac{4R_5(r)}{(R_5(r)-|z|)^2}T(R_5(r),f) \leq 8e\frac{T(r,f)}{r}[\log T(r,f)]^4 \tag{3.10}$$

for large $r \notin E$. To estimate the sum on the right hand side of (3.9) we apply Lemma 2.2, part (ii), with

$$H = \frac{r}{8[\log T(r,f)]^2} \tag{3.11}$$

and conclude that

$$\sum_{|z_j|\leq R_5(r)} \frac{1}{|z-z_j|} \leq \frac{n(R_5(r),0)(1+\log n(R_5(r),0))}{H} \tag{3.12}$$

outside a union of disks whose sum of radii is at most $2H$. Let P be the set of all $s > 0$ such that $\{z \in \mathbb{C} : |z| = s\}$ intersects the union of these disks. Then length $P \leq 4H$. Thus $F_r := [R_1(r), R_3(r)] \setminus P$ satisfies (3.5), and (3.12) holds for $|z| \in F_r$.

From (3.1) and (3.3) we can deduce that

$$\begin{aligned} n(R_5(r),0) &\leq \frac{T(R_6(r),f)}{\log(R_6(r)/R_5(r))} \\ &\leq 2T(R_6(r),f)[\log T(r,f)]^2 \leq 2eT(r,f)[\log T(r,f)]^2 \end{aligned} \tag{3.13}$$

for large $r \notin E$. Combining this with (3.11) and (3.12) we obtain

$$\begin{aligned} &\sum_{|z_j|\leq R_5(r)} \frac{1}{|z-z_j|} \\ &\leq \frac{16eT(r,f)[\log T(r,f)]^4\left(1+\log\left(2eT(r,f)[\log T(r,f)]^2\right)\right)}{r} \\ &\leq \frac{T(r,f)[\log T(r,f)]^6}{r} \end{aligned} \tag{3.14}$$

if $|z| \in F_r$ and r is large. Now (3.6) follows from (3.9), (3.10) and (3.14).

In order to prove (3.8) we use part (i) of Lemma 2.2 with

$$H = \frac{r}{2T(r,f)^{\eta/2}}. \tag{3.15}$$

This yields l disks $D(u_k, s_k)$ satisfying (3.7) such that

$$\sum_{|z_j|\le R_5(r)} \frac{1}{|z-z_j|} \le \frac{2n(R_5(r),0)}{H} \quad \text{for } z \notin \bigcup_{k=1}^{l} D(u_k, s_k), \tag{3.16}$$

with $l \le n(R_5(r),0)$. Combining (3.13), (3.15) and (3.16) we find that

$$\sum_{|z_j|\le R_5(r)} \frac{1}{|z-z_j|} \le \frac{8eT(r,f)[\log T(r,f)]^2 T(r,f)^{\eta/2}}{r}$$

if $z \notin \bigcup_{k=1}^{l} D(u_k, s_k)$. This, together with (3.9) and (3.10), implies (3.8). $\square$

Lemma 3.2. *Let F_r be as in Lemma 3.1 and let $\delta > 0$. If r is sufficiently large, then for each $s \in F_r$ there exists a closed subset J_s of $[0, 2\pi]$ with*

$$\text{length } J_s \ge \frac{1}{T(r,f)^{\delta}} \tag{3.17}$$

such that if $\theta \in J_s$, then

$$|f(se^{i\theta})| \ge \sqrt{M(r,f)} \tag{3.18}$$

and

$$\left|\frac{f'(se^{i\theta})}{f(se^{i\theta})}\right| \ge \frac{T(r,f)^{1-\delta}}{r}. \tag{3.19}$$

Proof. First we consider the case that

$$\log L(s,f) = \min_{|z|=s} \log|f(z)| \le \frac{1}{2}\log M(s,f). \tag{3.20}$$

Then there exists z_1 and z_2 with $|z_1| = |z_2| = s$ satisfying

$$\log|f(z_1)| = \frac{1}{2}\log M(s,f) \quad \text{and} \quad \log|f(z_2)| = \log M(s,f)$$

while

$$\frac{1}{2}\log M(s,f) \le \log|f(z)| \le \log M(s,f)$$

on one of the two arcs between z_1 and z_2. With $z_1 = se^{i\theta_1}$ and $z_2 = se^{i\theta_2}$ we may assume without loss of generality that $\theta_1 < \theta_2$ and

$$\frac{1}{2}\log M(s,f) \le \log|f(se^{i\theta})| \le \log M(s,f)$$

for $\theta_1 \le \theta \le \theta_2$. Let J_s be the set of all $\theta \in [\theta_1, \theta_2]$ for which (3.19) holds. Since (3.18) holds for all $\theta \in [\theta_1, \theta_2]$ and thus in particular for $\theta \in J_s$, we only have to estimate the length of J_s. By the choice of z_1 and z_2 we have

$$\begin{aligned} \frac{1}{2}\log M(s,f) &= \log|f(z_2)| - \log|f(z_1)| = \operatorname{Re}\left(\int_{z_1}^{z_2} \frac{f'(z)}{f(z)}dz\right) \\ &\le \int_{z_1}^{z_2} \left|\frac{f'(z)}{f(z)}\right| |dz| = s\int_{\theta_1}^{\theta_2} \left|\frac{f'(se^{i\theta})}{f(se^{i\theta})}\right| d\theta. \end{aligned} \tag{3.21}$$

Now

$$
\begin{aligned}
s\int_{\theta_1}^{\theta_2}\left|\frac{f'(se^{i\theta})}{f(se^{i\theta})}\right|d\theta &= s\int_{J_s}\left|\frac{f'(se^{i\theta})}{f(se^{i\theta})}\right|d\theta + s\int_{[\theta_1,\theta_2]\setminus J_s}\left|\frac{f'(se^{i\theta})}{f(se^{i\theta})}\right|d\theta \\
&\le s\frac{T(r,f)[\log T(r,f)]^7}{r}\,\text{length}\,J_s + 2\pi s\frac{T(r,f)^{1-\delta}}{r} \\
&\le 2T(r,f)[\log T(r,f)]^7\,\text{length}\,J_s + 4\pi\,T(r,f)^{1-\delta},
\end{aligned}
\tag{3.22}
$$

where we used $s \le R_3(r) \le 2r$ in the last inequality. Since $T(r,f) \le \log M(r,f)$ we have

$$4\pi\,T(r,f)^{1-\delta} \le \frac{1}{4}\log M(r,f)$$

for large r and this, together with (3.21) and (3.22), yields

$$\frac{1}{4}\log M(r,f) \le 2T(r,f)[\log T(r,f)]^7\,\text{length}\,J_s.$$

We deduce that

$$\text{length}\,J_s \ge \frac{1}{8[\log T(r,f)]^7}$$

and thus obtain (3.17).

Now we consider the case that (3.20) does not hold. Then (3.18) holds for all $\theta \in [0,2\pi]$. Let J_s be the subset of all $\theta \in [0,2\pi]$ for which (3.19) holds. By the argument principle, we have

$$n(s,0) = \frac{1}{2\pi i}\int_{|z|=s}\frac{f'(z)}{f(z)}dz \le \frac{s}{2\pi}\int_0^{2\pi}\left|\frac{f'(se^{i\theta})}{f(se^{i\theta})}\right|d\theta.$$

The same argument as in (3.22) now yields

$$2\pi\,n(s,0) \le 2T(r,f)[\log T(r,f)]^7\,\text{length}\,J_s + 4\pi\,T(r,f)^{1-\delta}. \tag{3.23}$$

Since we are assuming that (3.20) does not hold, we have $|f(re^{i\theta})| \ge 1$ for $0 \le \theta \le 2\pi$ and thus $m(r,1/f) = 0$, where $m(r,\cdot)$ denotes the Nevanlinna proximity function. Nevanlinna's first fundamental theorem, together with the assumption that $f(0) = 1$, yields

$$T(r,f) = N(r,0) \le N(1,0) + n(r,0)\log r$$

and thus

$$2\pi\,n(s,0) \ge 2\pi\,n(r,0) \ge \frac{T(r,f)}{\log r} \ge 2T(r,f)^{1-\delta/2} \tag{3.24}$$

for large r by (2.2). Combining (3.23) and (3.24) we obtain

$$T(r,f)^{1-\delta/2} \le 2T(r,f)[\log T(r,f)]^7\,\text{length}\,J_s,$$

from which (3.17) easily follows. □

For small $\delta > 0$ and large $r \notin E$ we consider the set

$$A(r) := \Big\{ z \in \mathbb{C} : R_1(r) \le |z| \le R_3(r), \\ |f(z)| \ge \sqrt{M(r,f)}, \ \left|\frac{f'(z)}{f(z)}\right| \ge \frac{T(r,f)^{1-\delta}}{r} \Big\}.$$

With the notation of Lemmas 3.1 and 3.2 we have

$$A(r) \supset \left\{ se^{i\theta} : s \in F_r, \ \theta \in J_s \right\}$$

and thus we can deduce from these lemmas that

$$\begin{aligned} \text{area}\, A(r) &\ge \text{length}(F_r) \inf_{s \in F_r} (s \,\text{length}(J_s)) \\ &\ge \frac{r^2}{[\log T(r,f)]^2 T(r,f)^{\delta}} \ge \frac{2r^2}{T(r,f)^{2\delta}} \end{aligned} \tag{3.25}$$

for large r. We put

$$\rho(r) := \frac{r}{T(r,f)^{1-2\delta}}$$

and

$$B(r) := A(r) \setminus \bigcup_{j=1}^{l} D\left(u_j, s_j + \rho(r)\right),$$

where $l, u_1, \dots, u_l, s_1, \dots, s_l$ are chosen according to Lemma 3.1, taking $\eta := 3\delta$ there.

We use the notation $\text{ann}(r,R) := \{z \in \mathbb{C} : r < |z| < R\}$ for an annulus with radii r and R. Given a constant $M > 1$ we have $M\rho(r) \le r/[\log T(r,f)]^2$ for large r and thus

$$D(b, M\rho(r)) \subset \text{ann}(r, R_4(r)) \quad \text{for } b \in A(r). \tag{3.26}$$

Hence

$$D(b,\rho(r)) \subset \text{ann}(r, R_4(r)) \setminus \bigcup_{j=1}^{l} D(u_j, s_j)$$

and thus

$$\left|\frac{f'(z)}{f(z)}\right| \le \frac{T(r,f)^{1+3\delta}}{r} \quad \text{if } b \in B(r) \text{ and } z \in D(b,\rho(r)) \tag{3.27}$$

by (3.8). In particular, f has no zeros in $D(b,\rho(r))$.

We want to show that the area of $B(r)$ is not much smaller than that of $A(r)$. In order to do so we note that (3.7), (3.13) and the definition of $\rho(r)$ yield

$$\begin{aligned}
\operatorname{area}\left(\bigcup_{j=1}^{l} D\left(u_j, s_j+\rho(r)\right)\right) &= \pi \sum_{j=1}^{l} \left(s_j+\rho(r)\right)^2 \\
&\leq 2\pi \sum_{j=1}^{l} \left(s_j^2+\rho(r)^2\right) \\
&\leq 2\pi \frac{r^2}{T(r,f)^{3\delta}} + 2\pi\rho(r)^2 n(R_5(r),0) \\
&\leq 2\pi \frac{r^2}{T(r,f)^{3\delta}} + 4\pi e \frac{r^2[\log T(r,f)]^2}{T(r,f)^{1-4\delta}} \\
&\leq \frac{r^2}{T(r,f)^{2\delta}}
\end{aligned}$$

for large r, provided $\delta < 1/6$. Combining this with (3.25) we obtain

$$\operatorname{area} B(r) \geq \operatorname{area} A(r) - \operatorname{area}\left(\bigcup_{j=1}^{l} D\left(u_j, s_j+\rho(r)\right)\right) \geq \frac{r^2}{T(r,f)^{2\delta}} \tag{3.28}$$

for large r.

Now let $m(r)$ be the maximal number of pairwise disjoint disks of radius $\rho(r)$ whose centers are in $B(r)$. Then $B(r)$ is contained in a union of $m(r)$ disks of radius $2\rho(r)$ and thus

$$\operatorname{area} B(r) \leq 4\pi m(r)\rho(r)^2 = 4\pi m(r) \frac{r^2}{T(r,f)^{2-4\delta}}.$$

for large r. Together with (3.28) we obtain

$$m(r) \geq \frac{1}{4\pi} T(r,f)^{2-6\delta} \geq T(r,f)^{2-7\delta}. \tag{3.29}$$

Recall that if $b \in B(r)$, for some large $r \notin E$, then f has no zeros in $D(b,\rho(r))$. Thus we can define a branch ϕ of the logarithm of f in $D(b,\rho(r))$; that is, there exists a holomorphic function $\phi : D(b,\rho(r)) \to \mathbb{C}$ such that $\exp\phi(z) = f(z)$ for $z \in D(b,\rho(r))$. Of course, the other branches of the logarithm of f are then given by $z \mapsto \phi(z) + 2\pi i n$ where $n \in \mathbb{Z}$.

For $a \in \mathbb{R}$ we will consider the domain

$$Q(a) := \{z \in \mathbb{C} : |\operatorname{Re} z - a| < 1, |\operatorname{Im} z| < 2\pi\}$$

Lemma 3.3. *Let $b \in B(r)$, with $r \notin E$ sufficiently large. Then there exists a branch ϕ of the logarithm of f defined in $D(b,\rho(r))$ and a subdomain U of $D(b,\rho(r))$ such that ϕ maps U bijectively onto $Q(\log|f(b)|)$.*

Proof. Let ϕ_0 be a fixed branch of the logarithm of f defined in $D(b,\rho(r))$. Define $h : \mathbb{D} \to \mathbb{C}$, $h(z) = \phi_0(b+\rho(r)z) - \phi_0(b)$. Then $h(0) = 0$ and

$$|h'(0)| = \rho(r)|\phi_0'(b)| = \rho(r)\left|\frac{f'(b)}{f(b)}\right| \geq \rho(r)\frac{T(r,f)^{1-\delta}}{r} = T(r,f)^{\delta}$$

by the definition of $\rho(r)$ and $A(r)$ and since $B(r) \subset A(r)$. For $k \in \{1,2,3\}$ we put $D_k := \{z \in \mathbb{C} : |\operatorname{Re} z| < 1, |\operatorname{Im} z - 12k\pi| < 4\pi\}$. Lemma 2.4 now implies that if $r \notin E$ is large enough, then there exists a subdomain V of $\mathbb{D}$ and $k \in \{1,2,3\}$ such that h maps V bijectively onto D_k. This implies that $D(b, \rho(r))$ contains a subdomain W which is mapped bijectively onto

$$\{z \in \mathbb{C} : |\operatorname{Re} z - \log|f(b)|| < 1, |\operatorname{Im} z - \operatorname{Im}\phi_0(b) - 12\pi k| < 4\pi\}$$

by ϕ_0. Choosing $n \in \mathbb{Z}$ such that $|2\pi n - \operatorname{Im}\phi_0(b) - 12k\pi| \le \pi$ we find that there exists a domain $U \subset W \subset D(b, \rho(r))$ which is mapped bijectively onto

$$\{z \in \mathbb{C} : |\operatorname{Re} z - \log|f(b)|| < 1, |\operatorname{Im} z - 2\pi n| < 2\pi\}$$

by ϕ_0. The branch ϕ of the logarithm given by $\phi(z) = \phi_0(z) - 2\pi i n$ now has the required property. □

We note that if U is as in Lemma 3.3, then

$$f(U) = \exp Q(\log|f(b)|) = \operatorname{ann}\left(\frac{1}{e}|f(b)|, e|f(b)|\right). \tag{3.30}$$

Now we fix some large $r_0 \notin E$, put $\rho_0 := \rho(r_0)$ and choose $b_0 \in B(r_0)$. Lemma 3.3 yields a branch ϕ_0 of the logarithm of f and a domain $U_0 \subset D(b_0, \rho_0)$ which is mapped bijectively onto $Q(\log|f(b_0)|)$ by ϕ_0. Since $|f(b_0)| \ge \sqrt{M(r_0, f)}$ we also have $|f(b_0)| \ge 2r_0$ and the interval $[|f(b_0)|, 2|f(b_0)|]$ contains a point r_1 which does not belong to E, provided r_0 is chosen large enough. Note that $r_1 \ge 2r_0$. We now choose $b_1 \in B(r_1)$ and with $\rho_1 := \rho(r_1)$ we find a domain $U_1 \subset D(b_1, \rho_1)$ which is mapped bijectively onto $Q(\log|f(b_1)|)$ by a branch ϕ_1 of the logarithm of f.

Inductively we thus obtain sequences (r_k), (ρ_k), (b_k), (U_k) and (ϕ_k) satisfying

$$r_k \in [\,|f(b_{k-1})|, 2|f(b_{k-1})|\,] \setminus E \subset [2r_{k-1}, \infty) \setminus E,$$

$\rho_k = \rho(r_k)$ and $b_k \in B(r_k)$ such that U_k is a subdomain of $D(b_k, \rho_k)$ and ϕ_k is a branch of the logarithm of f which has the property that $\phi_k : U_k \to Q(\log|f(b_k)|)$ is bijective.

For large r_0 we have $R_4(r_k) \le 5r_k/4 \le 5|f(b_k)|/2$ for all k and thus

$$\begin{aligned} D(b_k, M\rho_k) &\subset \operatorname{ann}(r_k, R_4(r_k)) \\ &\subset \operatorname{ann}\left(|f(b_{k-1})|, \frac{5}{2}|f(b_{k-1})|\right) \subset \exp Q(\log|f(b_{k-1})|) \end{aligned} \tag{3.31}$$

by (3.26) and (3.30). Hence there exists a branch L of the logarithm which maps $D(b_k, M\rho_k)$ into $Q(\log|f(b_{k-1})|)$. Then $\psi_k := \phi_{k-1}^{-1} \circ L$ is a branch of the inverse function of f which maps $D(b_k, M\rho_k)$ into U_{k-1}.

We put

$$V_k := (\psi_1 \circ \psi_2 \circ \ldots \circ \psi_k)\left(\overline{D}(b_k, \rho_k)\right).$$

Then V_k is compact and $V_{k+1} \subset V_k$. Thus $\bigcap_{k=1}^{\infty} V_k \ne \emptyset$. We will show that this intersection contains only one point.

In order to do so we note that since $\psi_k : D(b_k, M\rho_k) \to U_{k-1}$ is univalent and $U_{k-1} \subset D(b_{k-1}, \rho_{k-1}) \subset D(\psi_k(b_k), 2\rho_{k-1})$, Koebe's one quarter theorem implies

that $2\rho_{k-1} \geq M\rho_k|\psi_k'(b_k)|/4$ and thus $|\psi_k'(b_k)| \leq 8\rho_{k-1}/(M\rho_k)$. The Koebe distortion theorem, applied with $\lambda = 1/M$, now yields

$$\sup_{z\in\overline{D}(b_k,\rho_k)} |\psi_k'(z)| \leq \frac{1+\lambda}{(1-\lambda)^3}|\psi_k'(b_k)| = \frac{M^2(M+1)}{(M-1)^3}|\psi_k'(b_k)| \leq \frac{8M(M+1)}{(M-1)^3}\frac{\rho_{k-1}}{\rho_k}.$$

Choosing $M = 20$ we obtain

$$\sup_{z\in\overline{D}(b_k,\rho_k)} |\psi_k'(z)| \leq \frac{1}{2}\frac{\rho_{k-1}}{\rho_k}.$$

If $K \subset \overline{D}(b_k, \rho_k)$ is compact, we thus have

$$\operatorname{diam}\psi_k(K) \leq \frac{1}{2}\frac{\rho_{k-1}}{\rho_k}\operatorname{diam} K,$$

where $\operatorname{diam} K$ denotes the diameter of K. Hence

$$\frac{\operatorname{diam}\psi_k(K)}{\rho_{k-1}} \leq \frac{1}{2}\frac{\operatorname{diam} K}{\rho_k}.$$

Inductively we obtain

$$\frac{\operatorname{diam} V_k}{\rho_0} = \frac{\operatorname{diam}\left(\psi_1\circ\psi_2\circ\ldots\circ\psi_k\right)\left(\overline{D}(b_k,\rho_k)\right)}{\rho_0} \leq \frac{1}{2^k}\frac{\operatorname{diam}\overline{D}(b_k,\rho_k)}{\rho_k} = \frac{1}{2^{k-1}}$$

and thus

$$\lim_{k\to\infty}\operatorname{diam} V_k = 0 \tag{3.32}$$

so that

$$\bigcap_{k=1}^{\infty} V_k = \{z_0\}$$

for some z_0.

It follows from the definition of V_k and (3.26) that

$$f^k(V_k) = \overline{D}(b_k,\rho_k) \subset \operatorname{ann}(r_k, R_4(r_k))$$

and hence that $z_0 \in I(f)$. Moreover,

$$\begin{aligned} f^{k+1}(V_k) &= f\left(\overline{D}(b_k,\rho_k)\right) \supset f(U_k) \\ &= \operatorname{ann}\left(\frac{1}{e}|f(b_k)|, e|f(b_k)|\right) \supset \operatorname{ann}\left(\frac{1}{e}r_{k+1}, \frac{e}{2}r_{k+1}\right). \end{aligned}$$

As $F(f)$ does not have multiply connected components,

$$\operatorname{ann}\left(\frac{1}{e}r_{k+1}, \frac{e}{2}r_{k+1}\right) \cap J(f) \neq \emptyset$$

for large k. Since $J(f)$ is completely invariant, we conclude that V_k intersects $J(f)$, and since $J(f)$ is closed, this yields that $z_0 \in I(f) \cap J(f)$.

In order to estimate the upper box dimension of $I(f)\cap J(f)$, we note that in the above construction of the sequences (r_k), (ρ_k), (b_k), (U_k) and (ϕ_k) we have $m(r_k)$ choices for the point $b_k \in B(r_k)$ such that the disks of radius ρ_k around these points are pairwise disjoint, with $m(r_k)$ satisfying (3.29). We fix $k \in \mathbb{N}$,

choose r_j, ρ_j, b_j, U_j and ϕ_j for $0 \leq j \leq k-1$ as before and denote by b_k^ν choices of b_k with the above property, with $1 \leq \nu \leq m_k := m(r_k)$

In other words, we take $r_k \in [|f(b_{k-1})|, 2|f(b_{k-1})|] \setminus E$ and $\rho_k := \rho(r_k)$ as before and choose $b_k^\nu \in B(r_k)$, where $1 \leq \nu \leq m_k$, such that

$$D(b_k^i, \rho_k) \cap D(b_k^j, \rho_k) = \emptyset \quad \text{for } i \neq j. \tag{3.33}$$

Then for $1 \leq \nu \leq m_k$ there exists a branch $\psi_k^\nu : D(b_k^\nu, 2\rho_k) \to U_{k-1}$ of the inverse function of f that is of the form $\psi_k^\nu = \phi_{k-1}^{-1} \circ L_\nu$ for some branch L_ν of the logarithm which maps $D(b_k^\nu, 2\rho_k)$ into $Q(\log|f(b_{k-1})|)$. For $a \in \mathbb{R}$ we put

$$P(a) := \left\{ z \in \mathbb{C} : \ 0 \leq \operatorname{Re} z - a \leq \log \frac{5}{2}, \ |\operatorname{Im} z| \leq \frac{3}{2}\pi \right\}.$$

In view of (3.31) we may actually assume that L_ν maps $D(b_k^\nu, 2\rho_k)$ into the compact subset $P(\log|f(b_{k-1})|)$ of $Q(\log|f(b_{k-1})|)$. Since ϕ_{k-1}^{-1} is univalent in $Q(\log|f(b_{k-1})|)$ we thus conclude that there exists an absolute constant $\alpha > 1$ such that

$$\frac{1}{\alpha} \leq \frac{|(\phi_{k-1}^{-1})'(\zeta)|}{|(\phi_{k-1}^{-1})'(z)|} \leq \alpha \quad \text{for } \zeta, z \in P(\log|f(b_{k-1})|). \tag{3.34}$$

An explicit upper bound for α could be determined from the Koebe distortion theorem, but we do not need such an estimate. Put

$$\Lambda_k^\nu := \psi_1 \circ \psi_2 \circ \ldots \circ \psi_{k-1} \circ \psi_k^\nu = \psi_1 \circ \psi_2 \circ \ldots \circ \psi_{k-1} \circ \phi_{k-1}^{-1} \circ L_\nu.$$

Since $\psi_1 \circ \psi_2 \circ \ldots \circ \psi_{k-1}$ is univalent in $D(b_{k-1}, 2\rho_{k-1})$, we deduce from (3.34) and the Koebe distortion theorem that there exists $\beta > 1$ such that

$$\frac{1}{\beta} \leq \frac{|(\Lambda_k^\nu)'(b_k^\nu)|}{|(\Lambda_k^1)'(b_k^1)|} \leq \beta \quad \text{for } 1 \leq \nu \leq m_k. \tag{3.35}$$

We put

$$V_k^\nu := \Lambda_k^\nu \left(\overline{D}(b_k^\nu, \rho_k) \right) \quad \text{and} \quad v_k^\nu := \Lambda_k^\nu(b_k^\nu).$$

As above we see that each V_k^ν contains a point of $I(f) \cap J(f)$.

Since Λ_k^ν is univalent in $D(b_k^\nu, 2\rho_k)$ we deduce fron the Koebe distortion theorem (with $\lambda = 1/2$) that

$$\overline{D}\left(v_k^\nu, \frac{4}{9}\sigma_k^\nu \right) \subset V_k^\nu \subset \overline{D}\left(v_k^\nu, 4\sigma_k^\nu \right) \tag{3.36}$$

where

$$\sigma_k^\nu := |(\Lambda_k^\nu)'(b_k^\nu)|\, \rho_k = \frac{\rho_k}{|(f^k)'(v_k^\nu)|}.$$

We put $\sigma_k := \sigma_k^1$. It follows from (3.32) and (3.36) that

$$\lim_{k \to \infty} \sigma_k = 0.$$

Using (3.35) and (3.36) we obtain

$$\overline{D}\left(v_k^\nu, \frac{4}{9\beta}\sigma_k \right) \subset V_k^\nu \subset \overline{D}\left(v_k^\nu, 4\beta\sigma_k \right) \tag{3.37}$$

Fix a square of sidelength σ_k centered at a point c and denote by N the cardinality of the set of all $\nu \in \{1, \dots, m_k\}$ for which V_k^ν intersects this square. It follows from (3.37) that if if V_k^ν intersects this square, then $V_k^\nu \subset D(c, (8\beta+1)\sigma_k)$. On the other hand, (3.37) also says that V_k^ν contains a disk of radius $4\sigma_k/(9\beta)$. Since the V_k^ν have pairwise disjoint interior by (3.33), we obtain

$$N\pi \left(\frac{4\sigma_k}{9\beta}\right)^2 \leq \pi \left((8\beta+1)\sigma_k\right)^2.$$

Thus $N \leq N_0 := \lfloor 81\beta^2(8\beta+1)^2/16 \rfloor$.

We now put a grid of sidelength σ_k over U_0. Then each square of this grid can intersect at most N_0 of the m_k domains V_k^ν. Recalling that each of the domains V_k^ν contains a point of $I(f) \cap J(f)$ we see that at least m_k/N_0 squares of our grid intersect $I(f) \cap J(f)$. We conclude that

$$\overline{\dim}_{\mathrm{B}}(U_0 \cap I(f) \cap J(f)) \geq \limsup_{k\to\infty} \frac{\log(m_k/N_0)}{-\log \sigma_k}. \tag{3.38}$$

By (3.29) we have

$$m_k = m(r_k) \geq T(r_k, f)^{2-7\delta}. \tag{3.39}$$

It remains to estimate σ_k. In order to do so we note that

$$(f^k)'(v_k^1) = \prod_{j=0}^{k-1} f'(f^j(v_k^1)). \tag{3.40}$$

Since

$$f^j(v_k^1) \subset U_j \subset D(b_j, \rho_j)$$

it follows from (3.27) that

$$\left| \frac{f'(f^j(v_k^1))}{f(f^j(v_k^1))} \right| \leq \frac{T(r_j, f)^{1+3\delta}}{r_j}.$$

This yields

$$\left| f'(f^j(v_k^1)) \right| \leq \frac{T(r_j, f)^{1+3\delta}}{r_j} \left| f^{j+1}(v_k^1) \right| \leq e\frac{T(r_j, f)^{1+3\delta}}{r_j} r_{j+1}. \tag{3.41}$$

We deduce from (3.40) and (3.41) that

$$\left|(f^k)'(v_k^1)\right| \leq \prod_{j=0}^{k-1} e\frac{T(r_j, f)^{1+3\delta}}{r_j} r_{j+1} = e^k \frac{r_k}{r_0} \left(\prod_{j=0}^{k-1} T(r_j, f)\right)^{1+3\delta}$$

Using the definition of ρ_k and σ_k we thus have

$$\sigma_k \geq \frac{r_0}{e^k T(r_k, f)^{1-2\delta} \left(\prod_{j=0}^{k-1} T(r_j, f)\right)^{1+3\delta}} \tag{3.42}$$

By construction, we have

$$\log r_{j+1} \geq \log |f(b_j)| \geq \frac{1}{2} \log M(r_j, f)$$

for $0 \le j \le k-1$. Given a large positive number q, we deduce from (1.2) that if r_0 is sufficiently large, then

$$\log M(r_{j+1}, f) \ge (2\log r_{j+1})^{q+1}$$

for $0 \le j \le k-1$. Combining the last two estimates with (3.4) we find that

$$T(r_j, f) \le \log M(r_j, f) \le 2\log r_{j+1} \le (\log M(r_{j+1}, f))^{1/(q+1)} \le T(r_{j+1}, f)^{1/q}$$

for large r_0. We conclude that

$$\prod_{j=0}^{k-1} T(r_j, f) \le T(r_k, f)^{\tau}, \tag{3.43}$$

where

$$\tau := \sum_{j=1}^{k} \left(\frac{1}{q}\right)^j \le \sum_{j=1}^{\infty} \left(\frac{1}{q}\right)^j = \frac{1}{q-1}.$$

For large q we have $\tau(1+3\delta) \le \delta$ and thus

$$\left(\prod_{j=0}^{k-1} T(r_j, f)\right)^{1+3\delta} \le T(r_k, f)^{\delta}.$$

We can also deduce from (3.43) that

$$T(r_k, f)^{\delta} \ge e^k$$

if r_0 is chosen large enough. Combining the last two estimates with (3.42), and assuming that $r_0 \ge 1$, we conclude that $\sigma_k \ge 1/T(r_k, f)$.

Together with (3.38) and (3.39) we thus find that

$$\overline{\dim}_{\mathrm{B}}(U_0 \cap I(f) \cap J(f)) \ge \limsup_{k\to\infty} \frac{(2-7\delta)\log T(r_k, f) - \log N_0}{\log T(r_k, f)} = 2 - 7\delta.$$

Since $\delta > 0$ was arbitrary, we obtain (1.9) for $U = U_0$. We may assume that the closure of U_0 does not intersect the exceptional set. As mentioned in the introduction, Theorem 1.1 now follows.

Remark 3.1. Theorem B has been extended to meromorphic functions with finitely many poles [31] and in fact to meromorphic functions with a logarithmic tract [10, Theorem 1.4]. It is conceivable that our result admits similar extensions.

4. The minimum modulus of entire functions with multiply connected Fatou components

Zheng [37] proved that if the Fatou set of a transcendental entire function f has a multiply connected component U, then there exist sequences (r_k) and (R_k) satisfying $\lim_{k\to\infty} r_k = \lim_{k\to\infty} R_k/r_k = \infty$ such that $\operatorname{ann}(r_k, R_k) \subset f^k(U)$ for large k. It is shown in [11] that one can actually take $R_k = r_k^c$ for some $c > 1$

and this is then used to show that if $F(f)$ has a multiply connected component, then there exists $C > 0$ such that

$$\log L(r,f) \geq \left(1 - \frac{C}{\log r}\right) \log M(r,f) \tag{4.1}$$

on some unbounded sequence of r-values.

Using Zheng's result [37] instead of [11] yields the following proposition referred to in the introduction. We include its short proof for completeness.

Proposition 4.1. *Let f be a transcendental entire function for which $F(f)$ has a multiply connected component. Then* (1.7) *holds.*

Proof. Zheng [37, Corollary 1] used hyperbolic geometry to prove that the hypothesis of the proposition implies (1.6). We will use the same idea and denote the hyperbolic distance of two points a, b in a hyperbolic domain V by $\lambda_V(a,b)$; see, e.g., [25, Section 2.2] for the properties of the hyperbolic metric that are used.

Let U be a multiply connected component of $F(f)$ and let (r_k) and (R_k) be as in Zheng's result mentioned above. Put $s_k = \sqrt{R_k r_k}$ and $U_k = f^k(U)$. Choose $|a_k| = |b_k| = s_k$ such that $|f(a_k)| = L(s_k, f)$ and $|f(b_k)| = M(s_k, f)$. Since $R_k/r_k \to \infty$ we easily see that $\lambda_{U_k}(a_k, b_k) \leq \lambda_{\text{ann}(r_k,R_k)}(a_k, b_k) \to 0$. Thus $\lambda_{U_{k+1}}(f(a_k), f(b_k)) \to 0$. Since $0, 1 \notin U_{k+1}$ for large k we obtain

$$\lambda_{\mathbb{C}\setminus\{0,1\}}(f(a_k), f(b_k)) \to 0 \tag{4.2}$$

as $k \to \infty$. As the density $\rho_{\mathbb{C}\setminus\{0,1\}}(z)$ of the hyperbolic metric in $\mathbb{C} \setminus \{0,1\}$ satisfies $\rho_{\mathbb{C}\setminus\{0,1\}}(z) \geq c/(|z|\log|z|)$ for some $c > 0$ and large $|z|$, we obtain

$$\lambda_{\mathbb{C}\setminus\{0,1\}}(f(a_k), f(b_k)) \geq c \int_{|f(a_k)|}^{|f(b_k)|} \frac{dt}{t \log t} = c \log \frac{\log M(s_k, f)}{\log L(s_k, f)}. \tag{4.3}$$

Now (1.7) follows from (4.2) and (4.3). □

Remark 4.1. An alternative way to deduce Proposition 4.1 from Zheng's result is via Harnack's inequality [26, p. 14]. This method was used by Hinkkanen [22, Lemma 2] and Rippon and Stallard [32, Lemma 5], and it is also used in [11].

With the notation as in the above proof, put $t_k = \log s_k = \log \sqrt{R_k/r_k}$ and define

$$u_k : \{z \in \mathbb{C} : |\operatorname{Re} z| < t_k\} \to \mathbb{R}, \quad u_k(z) = \log |f(s_k e^z)|.$$

We may assume that $|f(z)| > 1$ for $z \in \text{ann}(r_k, R_k) \subset f^k(U)$ so that u_k is a positive harmonic function. Choose $y_1, y_2 \in \mathbb{R}$ with $|y_1 - y_2| \leq \pi$ such that $u(iy_1) = \log L(s_k, f)$ and $u(iy_2) = \log L(s_k, f)$. By Harnack's inequality we have

$$u(iy_2) \leq \frac{t_k + \pi}{t_k - \pi} u(iy_1) = (1 + o(1))u(iy_1)$$

as $k \to \infty$, and (1.7) follows.

Remark 4.2. It follows from a result of Fenton ([17], see also [13]) that if

$$p := \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log\log r} < \infty \tag{4.4}$$

and $\varepsilon > 0$, then

$$\log L(r,f) \geq \log M(r,f) - (\log r)^{p-2+\varepsilon} \tag{4.5}$$

on some sequence of r-values tending to ∞. Hence

$$\log L(r,f) \geq \log M(r,f) - \frac{\log M(r,f)}{(\log r)^{2-2\varepsilon}} \tag{4.6}$$

on such a sequence. Choosing $\varepsilon < 1/2$ we see that if

$$\lim_{r\to\infty} \left(1 - \frac{\log L(r,f)}{\log M(r,f)}\right) \log r = \infty, \tag{4.7}$$

then (4.6) and hence (4.4) cannot hold and thus (1.2) holds. The result of [11] quoted before Proposition 4.1 shows that if (4.7) holds, then $F(f)$ has no multiply connected components. Thus we obtain the following consequence of Theorem 1.1.

Corollary 4.1. *Let f be a transcendental entire function satisfying* (4.7). *Then* $\dim_{\mathrm{P}}(I(f) \cap J(f)) = 2$.

Finally we mention that it was actually shown in [11] and [17] that (4.1) and (4.5) hold on sets or r-values of a certain size. This could be used to further strengthen the statement of Corollary 4.1.

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